B.Sc. (Maths) part III

paper – VI

Topic :- Internal Direct product

(Group theory)

Internal Direct product

Def:- If H K be any two Sub groups of a group G then G is said to be internal product of H and K ie $G = H$ if ① Every element of H ~~commutes~~ Commutes with every element of K ⑪ Every element of G is uniquely expressible as a product of an element of K

Theorem ① If a Group G is the internal direct product of its sub group H and K then

① H and K have only the identity in common

⑪ G is isomorphic to External direct product of H by K ie. $G = H \times K$

Proof ① Consider $x \in H \cap K$ so that $x \in H$ and $x \in K$ and $x^{-1} \in H$, $x^{-1} \in K$ because both H and K are Sub group

Again $g \in G$ can be uniquely Expressed as $g = hk$: $h \in H, k \in K$ as $G = H \times K$ — ①
and $g = (hx)(x^{-1}k)$: $hx \in H, x^{-1}k \in K$ — ②

By defination of $G = H \times K$, each element of G can be uniquely Expressed as the product of an element of K

Thus from (1) and (2) we conclude that $hx = h$ and $x^{-1}k = k$

$\therefore hx = h \Rightarrow x = e$

Therefore e the identity is the only element common to H and K ie $e \in H \cap K$

(II) Suppose that mapping $\phi : G \to H$ is defined by

$\phi(g) = \phi(hk) = (h, k) \ \forall \ g \in G$

ϕ is one-one : Let $\phi(g_1) = \phi(g_2)$

$\Rightarrow (h_1, k_1) = (h_2, k_2)$

$\Rightarrow h_1 = h_2$ and $k_1 = k_2$

$\Rightarrow h_1 k_1 = h_2 k_2$

$\Rightarrow g_1 = g_2$

$\Rightarrow \phi$ is one-one

ϕ is onto :- Suppose (h, k) be any element of $H \times K$ then $hk \in G$ Also $\phi(hk) = (h, k)$ by def.

$\Rightarrow \phi$ is onto

ϕ preserves the Composition

$\phi(g_1 g_2) = \phi(h_1 k_1, h_2 k_2)$

$= \phi(h_1 h_2 k_1 k_2)$

because every element of H
commutes with every element of K
$\therefore \phi(g_1 \cdot g_2) = (h_1 h_2, k_1 k_2)$
$= (h_1 k_1)(h_2 k_2)$
$= \phi(h_1 k_1)\, \phi(h_2 k_2)$
$= \phi(g_1)\, \phi(g_2)$
$\therefore \phi : G \to H \times K$ is one-one onto
and preserves the group composition
therefore it is an isomorphism.